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INVARIANT DESCRIPTION OF FIXED-WING UNMANNED AERIAL VEHICLE MOTION IN VERTICAL PLANE

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ABSTRACT: *This study presents an invariant description of fixed-wing unmanned aerial vehicle (UAV) motion within a vertical plane, yielding closed-form expressions for the invariants. Through integration specific to this UAV motion, invariants are determined as functions of the flight-path angle. The simulation results illustrate that the invariants remain unaffected by changes in heading angle, demonstrating consistent values regardless of heading direction. Additionally, the results show that an increase in bank angle correlates with a reduced rate of change in the invariants concerning flight-path angle. The obtained results offer insights into parameter evaluation and the development of invariant-based control and guidance methodologies.*

KEYWORDS: *Fixed-wing UAV, instantaneous screw motion, invariant description of motion.*

1. INTRODUCTION

Dynamic modeling, simplifications in the procedure, and kinematics are important factors in controlling a dynamical system. In this paper, the instantaneous screw motion (ISM) concept will be discussed along with the fixed-wing unmanned aerial vehicle (UAV) dynamics under several assumptions.

Unlike the traditional representation of a motion in terms of decoupled translation and rotation, Gulio Mozzi proved in 1763 that any general rigid body motion can be described by an instantaneous screw axis (ISA) [1,2]. One of the applications of the ISM concept in rigid body kinematics is the invariant description of rigid body motion that is presented by J. Schutter [3]. J. Angeles described the automatic computation of screw parameters of rigid body motion for both finitely and infinitesimally separated positions [5,6]. A deep study on screw calculus and its applications in mechanics in terms of vector analysis is provided in Ref. [7].

Unmanned aerial vehicles are flying vehicles with no human pilot onboard. Over the last two decades, they have received considerable attention due to their capability of carrying out a number of tasks [8]. Particularly, the demand for highly accurate fixed-wing UAVs has increased significantly in the recent years. The idea of utilizing the ISM invariants in control and guidance problems was first proposed by D. Azimov in 2013, and the advantages and potential uses of this concept in guidance techniques have been explained [8].

This paper is organized and explained in the following steps. First, the equations of motion of a fixed-wing UAV dynamic model and their first integrals will be provided in Section 2. Second, the expressions for the ISM invariants will be shown for a particular case of a fixed-wing UAV motion. In Section 3, special cases of the motion will be discussed including the rotational and pure translational motions. Next, the simulation results will be shown and the key points will be discussed. The last section provides the conclusions from the study.

2. EQUATIONS OF MOTION

The equations of motion of an aircraft can be defined using two frames: the inertial frame and the body frame. The inertial frame is fixed on the ground at sea level and denoted as $Exyh$. The body frame is denoted $Be_1e_2e_3$ and fixed on the aircraft's center of gravity (COG) with the velocity vector pointing in e_1 - direction.

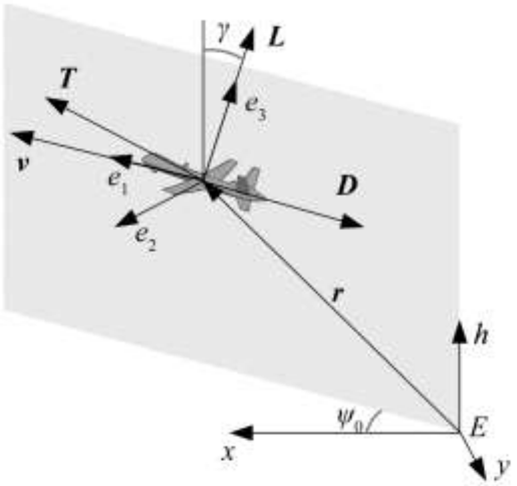

Fig. 1. Initial and body frame.

If the sideslip angle is zero (i.e. $\beta = 0$) and the bank angle is constant (i.e. $\phi = \phi_0$), then the atmospheric flight equations can be obtained as [9]

$$
\begin{aligned}
\dot{x} &= v\cos\gamma\cos\psi,\\
\dot{y} &= v\cos\gamma\sin\psi,\\
\dot{h} &= v\sin\gamma\,,\\
\dot{v} &= \tfrac{g_0}{m}(T\cos\alpha - D) - g_0\sin\gamma,\\
\dot{\gamma} &= \frac{g_0}{Wv}(T\sin\alpha + L) - \frac{g_0}{v}\cos\gamma,\\
\dot{\psi} &= \frac{g_0}{Wv\cos\gamma}(T\sin\alpha + L)\sin\varphi_0,
\end{aligned}
\tag{2.1}
$$

where x, y, h - coordinates, v - velocity magnitude, γ - flight path angle, ψ - heading angle, ϕ_0 - bank angle, α - angle of attack, g_0 - magnitude of gravitational acceleration, T - thrust, L - lift, D - drag, W - weight, C - specific fuel consumption. The lift and drag are defined to be the components of the resultant aerodynamic force perpendicular and parallel to the velocity vector [9]:

$$L = \tfrac{1}{2}C_L\rho S v^2,\;\; D = \tfrac{1}{2}C_D\rho S v^2, \tag{2.2}$$

where C_L, C_D - lift and drag coefficients, ρ - the density of the atmosphere at the altitude of the aircraft, and S is the wing platform area.

Assuming that the change of weight is negligible and the following expressions are constant over the time during the flight

$$\frac{g_0}{W}(T\cos\alpha - D) = c_1,\ \ \frac{g_0}{W}(T\sin\alpha + L) = c_2,\ \ c_1,\ c_2 - const. \tag{2.3}$$

Eqs. (2.1) can be rewritten in the following form [10,11]

$$\begin{aligned} \dot{x} &= v\cos\gamma\cos\psi\,, \\ \dot{y} &= v\cos\gamma\,\sin\psi\,, \\ \dot{h} &= v\sin\gamma\,, \\ \dot{v} &= c_1 - g_0\sin\gamma\,, \\ \dot{\gamma} &= \frac{1}{v}(c_2\cos\varphi_0 - g_0)\cos\gamma\,, \\ \dot{\psi} &= \frac{c_2\sin\varphi_0}{v\cos\gamma}. \end{aligned} \tag{2.4}$$

The integrals of Eqs. (2.4) can be obtained in terms of γ, assuming that γ is an independent variable [11]:

$$\begin{aligned} x(\gamma) &= \int \frac{v^2(\lambda)\sin\lambda\cos\psi\,d\lambda}{a+b\sin\lambda} + \eta_1, \\ y(\gamma) &= \int \frac{v^2(\lambda)\sin\lambda\sin\psi\,d\lambda}{a+b\sin\lambda} + \eta_2, \\ h(\gamma) &= Q(\gamma)\exp\left[\frac{4A}{d_1}\arctan\frac{a\tan\bar{\lambda}+1}{d_1}\right] + \eta_3, \\ v(\gamma) &= \eta_4(a+b\sin\lambda)^{-1}\exp\left[\frac{2A}{d_1}\arctan\frac{a\tan\bar{\lambda}+1}{d_1}\right], \\ \psi(\gamma) &= \tan\varphi_0\ln\left(\tan\bar{\lambda}\right) + \frac{2g_0}{d_1}\tan\varphi_0\arctan\left(\frac{a\tan\bar{\lambda}+1}{d_1}\right) + \eta_5, \end{aligned} \tag{2.5}$$

where η_{1-5}, A - integration constants, $a = c_2\cos\phi_0, b = -g_0,\ \lambda = \gamma + \frac{\pi}{2}, \bar{\lambda} = \frac{\lambda}{2}, Q(y)$ - function of γ [11].

For simplicity purposes, the heading angle will be considered constant, i.e the motion is in a vertical plane.

3. CALCULATIONS OF INVARIANTS

The motion of a rigid body can be described using six independent instantaneous screw motion (ISM) invariants [3]. The expressions for these six invariants are obtained in the following form [3]:

$$i_1 = f_1(\bar{\omega}),\ i_2 = f_2(\bar{\omega}, \bar{v}), i_3 = f_3(\bar{\omega}, \dot{\bar{\omega}}), i_4 = f_4(\bar{\omega}, \dot{\bar{\omega}}, \bar{v}, \dot{\bar{v}}), i_5 = f_5\left(\bar{\omega}, \dot{\bar{\omega}}, \ddot{\bar{\omega}}\right),$$

$$i_6 = f_6\left(\bar{\omega}, \dot{\bar{\omega}}, \ddot{\bar{\omega}}, \bar{v}, \dot{\bar{v}}, \ddot{\bar{v}}\right), \tag{3.1}$$

where $\bar{\omega}$ and $\bar{v}$ are the angular and translational velocity vectors respectively.

For the motion of an aircraft in a vertical plane, these vectors are obtained as [10]

$$\bar{\omega} = \frac{\dot{\bar{r}}\times\ddot{\bar{r}}}{|\dot{\bar{r}}|^2},\ v = \dot{\bar{r}}, \tag{3.2}$$

where $\bar{r}$ - position vector of an aircraft.

The expressions for invariants can be derived as

$$i_1 = |\bar{\omega}| = \left(\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|\right) / \left(\left|\dot{\bar{r}}\right|^2\right),\ i_2 = \frac{(\bar{v}\cdot\bar{\omega})}{|\bar{\omega}|} = 0,\ i_3 = \frac{|\bar{\omega}\times\dot{\bar{\omega}}|}{|\bar{\omega}|^2} = \frac{\left|\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right|}{\left|\dot{\bar{r}}\right|^2\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|},$$

$$i_4 = \bar{e}_y \cdot \dot{\bar{p}},\ i_5 = \frac{\left|\left[\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right]\times\left[\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\ddddot{\bar{r}}+\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right]\right|}{\left|\dot{\bar{r}}\right|^4\left|\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right|},\ i_6 = \dot{\bar{p}} \cdot \bar{e}_x - \dot{p}_2, \tag{3.3}$$

where

$$\bar{p} = \frac{\bar{\omega}\times\bar{v}}{|\bar{\omega}|^2},\ \dot{\bar{p}} = \left[\frac{\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\times\dot{\bar{r}}+\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\ddot{\bar{r}}}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|} - \frac{2\left|\dot{\bar{r}}\right|^4\left\{\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right\}\cdot\left\{\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\dot{\bar{r}}\right\}}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|^4}\right], \bar{e}_x = \frac{\dot{\bar{r}}\times\ddot{\bar{r}}}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|},$$

$$\bar{e}_y = \frac{\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)}{\left|\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right|}, \bar{e}_z = \frac{\left[\dot{\bar{r}}\times\ddot{\bar{r}}\right]\times\left[\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right]}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|\cdot\left|\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right|}, p_2 = -\frac{\bar{p}\cdot\bar{e}_z}{\omega_2}, \omega_2 = \frac{\left|\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)\times\left(\dot{\bar{r}}\times\dddot{\bar{r}}\right)\right|}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|^2}.$$

From Eqs. (2.4), the derivatives of the position vector can be written in the following form:

$$\dot{r} = \begin{bmatrix} v\cos\gamma\cos\psi_0 \\ v\cos\gamma\sin\psi_0 \\ v\sin\gamma \end{bmatrix}, \tag{3.4}$$

$$\dot{\bar{r}} = \begin{bmatrix} \dot{v}\cos\gamma\cos\psi_0 - v\dot{\gamma}\sin\gamma\cos\psi_0 \\ \dot{v}\cos\gamma\sin\psi_0 - v\dot{\gamma}\sin\gamma\sin\psi_0 \\ \dot{v}\sin\gamma + v\,\dot{\gamma}\cos\gamma \end{bmatrix} = \begin{bmatrix} (c_1\cos\gamma - c_2\cos\phi_0\sin\gamma)\cos\psi_0 \\ (c_1\cos\gamma - c_2\cos\phi_0\sin\gamma)\sin\psi_0 \\ c_1\sin\gamma + c_2\cos\phi_0\cos\gamma - g_0 \end{bmatrix}. \quad (3.5)$$

Substitution of Eqs.(3.4, 3.5) into Eqs. (3.2) allows us to get the invariants as the functions of γ (flight-path angle) and α (angle of attack).

4. SPECIAL CASES

In this section, some special cases of aircraft's motion will be studied. The assumptions made above are considered to be invalid for this section. The roll, pitch, and yaw angles are denoted by ϕ, θ, and ψ respectively.

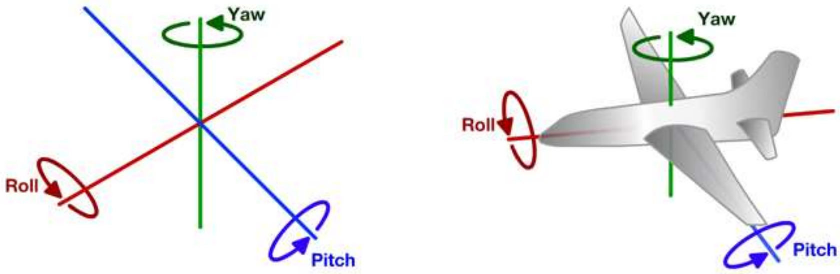

Fig. 2. Roll, Pitch and Yaw angles [14]

4.1.Rotational motion

4.1.1. Roll maneuver: $\dot{\phi} \neq 0,\ \dot{\theta} = 0,\ \dot{\psi} = 0$

Consider the motion that the aircraft performs a roll maneuver. In this case, the first and the second invariants can be written in terms of bank and sideslip angles. The other four invariants would be zero:

$$i_1 = |\bar{\omega}| = \dot{\phi},\quad i_2 = |\dot{\bar{r}}| = v = \eta_4(a + b\sin\lambda)^{-1}\exp\left[\frac{2A}{d_1}\arctan\frac{a\tan\bar{\lambda} + b}{d_1}\right],$$

$$i_3 = i_4 = i_5 = i_6 = 0. \quad (4.1)$$

4.1.2. Pitch maneuver: $\dot{\phi} = 0,\ \dot{\theta} \neq 0,\ \dot{\psi} = 0$

In this case, the aircraft performs a pitch maneuver, and the invariants will take the following form:

$i_1 = \dot{\theta}\,|\cos\phi_0 - \sin\phi_0|,\ i_2 = \dot{y}, \qquad i_3 = i_4 = i_5 = i_6 = 0,$

(4.2)

where ϕ_0 is a constant bank angle.

4.1.3. *Yaw maneuver:* $\dot{\phi} = 0,\ \dot{\theta} = 0,\ \dot{\psi} \neq 0$

For this case of aircraft's motion, the rotational and the translational velocity vectors can be written as [13]

$\overline{\omega} = [0\ 0\ (\cos\theta_0\cos\phi_0 + \cos\theta_0\sin\phi_0 - \sin\theta_0)]^T,\ \ \overline{v} = [\dot{x}\ \dot{y}\ \dot{z}]^T,$

(4.3)

where θ_0 is a constant pitch angle.

The invariants will be as follows

$i_1 = \dot{\psi}\,|\cos\theta_0\cos\phi_0 + \cos\theta_0\sin\phi_0 - \sin\theta_0|,\ i_2 = \dot{z},\ i_3 = i_4 = i_5 = i_6 = 0.$

(4.4)

4.2. Translational motion

When the aircraft has only the translational velocity, the screw axis would be in the direction of this velocity vector, and the only nonzero invariant would be the invariant 2. Depending on the choice of the $\overline{e}_x$ unit vector, it could be either positive or negative:

$i_1 = 0,\ \ i_2 = v = \eta_4(a + b\sin\lambda)^{-1}\exp\left[\frac{2A}{d_1}\arctan\frac{a\tan\overline{\lambda}+b}{d_1}\right],\ \ i_3 = i_4 = i_5 = i_6 = 0.$

(4.5)

5. SIMULATIONS

5.1. Simulation setup

Since the weight change is considered negligible, c_1 and c_2 constants in Eq. (2.3) represent the accelerations along the wind axis and the lift axis respectively.

In most cases, the lift acceleration is greater than the gravitational acceleration, meaning that $(c_2\cos\phi_0)^2 > g_0^2$. Without loss of generality, the following values for the constants can be chosen to simulate the obtained results [9]:

$$m = 40kg,\ g_0 = 9.81m/\sec^2\quad,\ S = 21.55,\quad K = 0.073,\ Cd_0 = 0.0223,$$

$$Ar = 5.1,\ \alpha_{0L} = 0.1,\ \alpha_T = 0.1,\ c_1 = 1,\ c_2 = 10,\ \psi_0 = \pi/4,\ \phi_0 = 0.$$

Matlab was used to create the diagrams.

5.2. Graphical relationship between parameters

An implicit, or possibly, explicit relationships between the parameters are of great interest. The profiles of the ISM invariants with respect to flight-path angle and angle of attack are shown in Figure 3. In addition, the invariants are tested by changing the constant bank angle. The invariants have also been tested by changing the heading angle and the bank angle. It turned out that the invariants don't depend on the heading angle if it is constant, i.e., the motion is on a vertical plane. Furthermore, it is seen that the rate of invariants with regard to the flight path angle reduces as the bank angle increases. Figure 3 represents the invariants with respect to the angle of attack and flight path angle considering the bank and heading angles constant. It can be seen that the first invariant (i_1) is proportional to the angle of attack, and increases parabolically as the flight-path angle increases. The second invariant is zero for any values of γ and α under the assumptions considered. The third invariant fluctuates in a small interval (nearly zero), and the interval becomes wider as γ increases. The 4th invariant fluctuates in the interval [-1000,1000], and the interval shrinks as γ increases. The fifth invariant changes between 0 and 1.5, and it is hard to evaluate the change of interval. The invariant 6 changes dramatically with random fluctuations, making it hard to evaluate its behavior.

6. CONCLUSIONS

The invariant description of a fixed-wing UAV motion in a vertical plane has been presented and the closed-form expressions for the invariants have been derived. Using the integrals obtained for this particular motion of a fixed-wing UAV, the invariants are found as the functions of the flight-path angle. An implicit relationship between the invariants and the parameters of flight dynamics (including the control parameters) has been established. The simulation results show that the invariants are independent of the heading angle which means that for any constant heading angle, the invariants are found to be the same. It is also shown that the increase in the bank angle decreases the rate of the invariants with respect to the flight-path angle. The results can be used to evaluate the parameters in control and guidance problems and develop invariant-based control and guidance systems.

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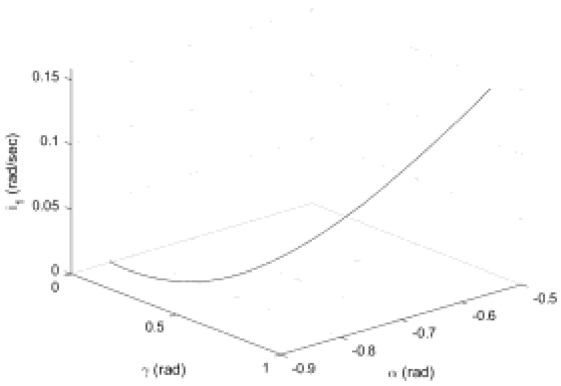
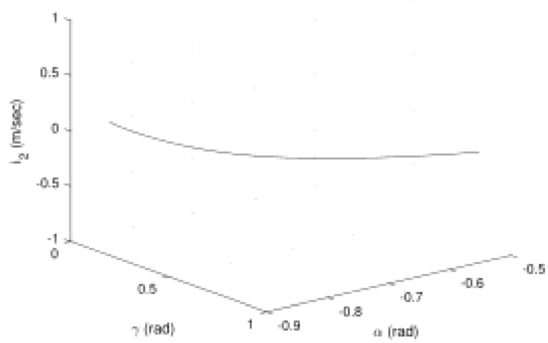


(a)



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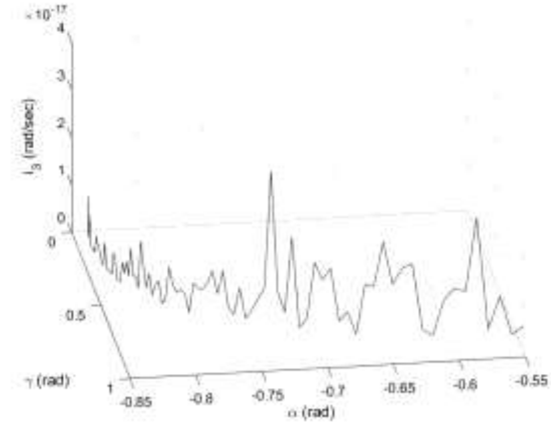

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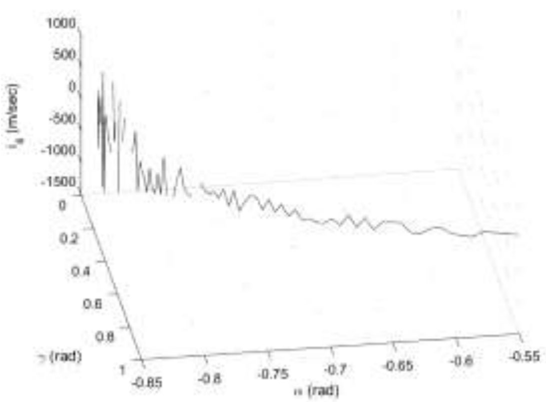

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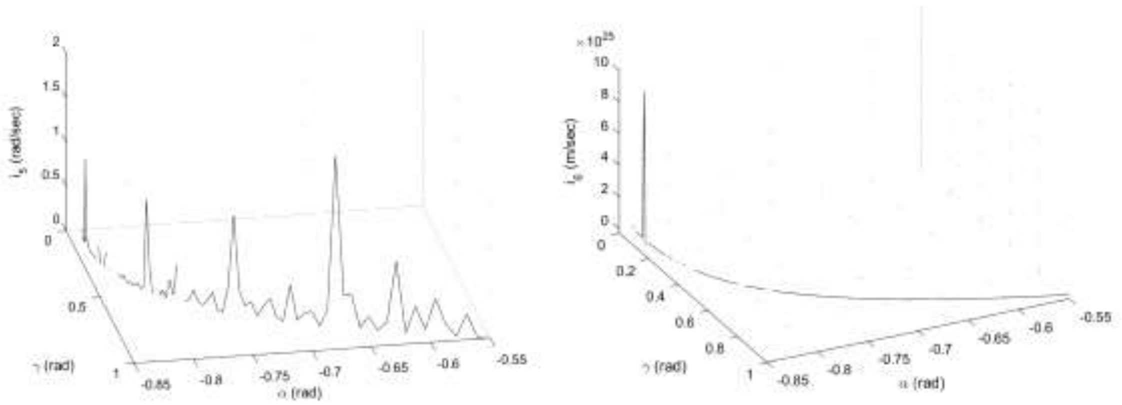

(e) (f)

Fig. 3. Invariants vs Angle of attack and Flight path angle

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